Making Static Pivoting Dependable
E. Jason Riedy
EECS Department, UC Berkeley
ejr@cs.berkeley.edu

1. Motivation
For sparse LU factorization, dynamic pivoting tightly couples symbolic and numerical computation. Dynamic structural changes limit parallel scalability. Demmel and Li use static pivoting in distributed SuperLU for performance, but intentionally perturbing the input may lead silently to erroneous results. Are there experimentally stable static pivoting heuristics that lead to a dependable direct solver? The answer is currently a qualified yes. Current heuristics fail on a few systems, but all failures are detectable.

2. Pivoting and Perturbation Heuristics
• Partial pivoting is a heuristic to limit element growth.
• Another approach: perturb pivots to keep divisors large.
• Small perturbations should not overly affect stability.

Pivoting Heuristics
Partial Pivoting
- Given: ratio \( u \geq 1 \).
- \( i_p = \arg\max_{i,j \neq j} |a(i,j)| \).
- If \( |a(i,j)| < T \cdot |a(j,j)| \), add \( n \cdot P \) to \( |a(i,j)| \) (preserving sign).

Static Pivoting
- Given: threshold \( T \), perturbation \( P \).

Perturbation Heuristics
- Norm-relative: Keep perturbations small relative to global norm. Used in SuperLU with parameter \( \rho = 2^{-17} \) machine precision \( \approx 2^{-28} \).
- Diagonal-relative: Sparsity-preserving orderings limit changes to the diagonal, so keep perturbations small relative to original diagonal.

Norm-Relative
- \( T = P = \rho \cdot \|A\| \).

Diagonal-Relative
- Save original diagonal entries in \( d \).
- At column \( j \), \( T = P = \rho \cdot |d(j)| \).

3. Weighted Bipartite Matchings
• To minimize the number of perturbations during factorization, we pre-pivot to place large entries on the diagonal.
• Parallelizable auction algorithms can maximize product or sum of the diagonal, summing produce less reliable pivots.
• Quantizing diagonal to integer saves around 10% of execution time with no noticeable effect on pivots.
• Approximating the objective improving time by 30% – 300%, also with little noticeable effect.

4. Iterative Refinement
• Newton’s method applied to \( Ax = b \).
• Can use extra precision in residual and temporary solution to obtain norm-wise and componentwise accuracy.
• See Yozo Hida’s poster for full details.
Negligible Incremental Cost: Extra-precise residuals approximately \( 25\text{min} \) fp operations in software, but no additional memory accesses. If the factoring produces a fill of over \( 12x \), then extra-precise residuals are no more expensive than each step’s solve.

5. Summary
Out of 231 large matrices from UF Collection (N from 2500 to 213 360):
• Extra-precise refinement failed to stabilize five matrices for the best norm-relative perturbations and two for diagonal relative (Zhao2, av41092).
• All failed from pivot growth > 10^6; simple to check.
• These systems were the only error estimate failures.
• Note: Only worried about condition numbers less than \( 10^{12} \) machine precision, depending on non-zeros per row / col.
• Little variation
Failures are rare and detectable. Careful scaling should reduce pivot growth in current failure cases.